

Multiple little q -Jacobi polynomials[★]

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Abstract

We introduce two kinds of multiple little q -Jacobi polynomials $p_{\vec{n}}$ with multi-index $\vec{n} = (n_1, n_2, \dots, n_r)$ and degree $|\vec{n}| = n_1 + n_2 + \dots + n_r$ by imposing orthogonality conditions with respect to r discrete little q -Jacobi measures on the exponential lattice $\{q^k, k = 0, 1, 2, 3, \dots\}$, where $0 < q < 1$. We show that these multiple little q -Jacobi polynomials have useful q -difference properties, such as a Rodrigues formula (consisting of a product of r difference operators). Some properties of the zeros of these polynomials and some asymptotic properties will be given as well.

Key words: q -Jacobi polynomials, basis hypergeometric polynomials, multiple orthogonal polynomials

1 Little q -Jacobi polynomials

Little q -Jacobi polynomials are orthogonal polynomials on the exponential lattice $\{q^k, k = 0, 1, 2, \dots\}$, where $0 < q < 1$. In order to express the orthogonality relations, we will use the q -integral

$$\int_0^1 f(x) d_q x = (1 - q) \sum_{k=0}^{\infty} q^k f(q^k), \quad (1.1)$$

(see, e.g., [2, §10.1], [5, §1.11]) where f is a function on $[0, 1]$ which is continuous at 0. The orthogonality is given by

$$\int_0^1 p_n(x; \alpha, \beta | q) x^k w(x; \alpha, \beta | q) d_q x = 0, \quad k = 0, 1, \dots, n-1, \quad (1.2)$$

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where

$$w(x; a, b|q) = \frac{(qx; q)_\infty}{(q^{\beta+1}x; q)_\infty} x^\alpha. \quad (1.3)$$

We have used the notation

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

In order that the q -integral of w is finite, we need to impose the restrictions $\alpha, \beta > -1$. The orthogonality conditions (1.2) determine the polynomials $p_n(x; \alpha, \beta|q)$ up to a multiplicative factor. In this paper we will always use monic polynomials and these are uniquely determined by the orthogonality conditions. The q -binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z|, |q| < 1, \quad (1.4)$$

(see, e.g., [2, §10.2], [5, §1.3]) implies that

$$\lim_{q \rightarrow 1} w(x; \alpha, \beta|q) = (1 - x)^\beta x^\alpha, \quad 0 < x < 1,$$

so that $w(x; \alpha, \beta|q)$ is a q -analog of the beta density on $[0, 1]$, and hence

$$\lim_{q \rightarrow 1} p_n(x; \alpha, \beta|q) = P_n^{(\alpha, \beta)}(x),$$

where $P_n^{(\alpha, \beta)}$ are the monic Jacobi polynomials on $[0, 1]$. Little q -Jacobi polynomials appear in representations of quantum $SU(2)$ [9], [10], and the special case of little q -Legendre polynomials was used to prove irrationality of a q -analog of the harmonic series and $\log 2$ [14]. Their role in partitions was described in [1]. A detailed list of formulas for the little q -Jacobi polynomials can be found in [8, §3.12], but note that in that reference the polynomial $p_n(x; a, b|q)$ is not monic and that $a = q^\alpha, b = q^\beta$. Useful formulas are the *lowering operation*

$$\mathcal{D}_q p_n(x; \alpha, \beta|q) = \frac{1 - q^n}{1 - q} p_{n-1}(x; \alpha + 1, \beta + 1|q), \quad (1.5)$$

where \mathcal{D}_q is the q -difference operator

$$\mathcal{D}_q f(x) = \begin{cases} \frac{f(x) - f(qx)}{(1 - q)x}, & \text{if } x \neq 0, \\ f'(0), & \text{if } x = 0, \end{cases} \quad (1.6)$$

and the *raising operation*

$$\begin{aligned} & \mathcal{D}_p[w(x; \alpha, \beta|q)p_n(x; \alpha, \beta|q)] \\ &= -\frac{1 - q^{n+\alpha+\beta}}{(1 - q)q^{n+\alpha-1}} w(x; \alpha - 1, \beta - 1|q) p_{n+1}(x; \alpha - 1, \beta - 1|q), \end{aligned} \quad (1.7)$$

where $p = 1/q$. Repeated application of the raising operator gives the *Rodrigues formula*

$$w(x; \alpha, \beta|q)p_n(x; \alpha, \beta|q) = \frac{(-1)^n(1-q)^n q^{\alpha n + n(n-1)}}{(q^{\alpha+\beta+n+1}; q)_n} \mathcal{D}_p^n w(x; \alpha + n, \beta + n|q). \quad (1.8)$$

A combination of the raising and the lowering operation gives a *second order q -difference equation*. The Rodrigues formula enables us to give an explicit expression as a basic hypergeometric sum:

$$p_n(x; \alpha, \beta|q) = \frac{x^n q^{n(n+\alpha)} (q^{-n-\alpha}; q)_n}{(q^{n+\alpha+\beta+1}; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, q^{-n-\alpha}, 1/x \\ q^{\beta+1}, 0 \end{matrix} \middle| q; q \right),$$

which by some elementary transformations can also be written as

$$\begin{aligned} p_n(x; \alpha, \beta|q) &= \frac{q^{(n+\alpha)n} (q^{-n-\alpha}; q)_n}{(q^{n+\alpha+\beta+1}; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1} \\ q^{\alpha+1} \end{matrix} \middle| q; qx \right) \\ &= \frac{q^{(n+\alpha)n} (q^{-n-\alpha}; q)_n}{(q^{n+\alpha+\beta+1}; q)_n} \sum_{k=0}^n \frac{(q^{-n}; q)_k (q^{n+\alpha+\beta+1}; q)_k}{(q^{\alpha+1}; q)_k} \frac{q^k x^k}{(q; q)_k}. \end{aligned} \quad (1.9)$$

2 Multiple orthogonal polynomials

Multiple orthogonal polynomials (of type II) are polynomials satisfying orthogonality conditions with respect to $r \geq 1$ positive measures [3] [4] [11, §4.3] [15]. Let $\mu_1, \mu_2, \dots, \mu_r$ be r positive measures on the real line and let $\vec{n} = (n_1, n_2, \dots, n_r) \in \mathbb{N}^r$ be a multi-index of length $|\vec{n}| = n_1 + n_2 + \dots + n_r$. The corresponding type II multiple orthogonal polynomial $p_{\vec{n}}$ is a polynomial of degree $\leq |\vec{n}|$ satisfying the orthogonality relations

$$\int p_{\vec{n}}(x) x^k d\mu_j(x) = 0, \quad k = 0, 1, \dots, n_j - 1, \quad j = 1, 2, \dots, r.$$

These orthogonality relations give $|\vec{n}|$ homogeneous equations for the $|\vec{n}| + 1$ unknown coefficients of $p_{\vec{n}}$. We say that \vec{n} is a normal index if the orthogonality relations determine the polynomial $p_{\vec{n}}$ up to a multiplicative factor. Multiple orthogonal polynomials of type I (see, e.g., [3] [11, §4.3] [4] [15]) will not be considered in this paper. Multiple little q -Jacobi polynomials are multiple orthogonal polynomials where the measures μ_1, \dots, μ_r are supported on the exponential lattice $\{q^k, k = 0, 1, 2, \dots\}$ and are all of the form $d\mu_i(x) = w(x; \alpha_i, \beta_i|q) d_q x$, where $w(x; \alpha, \beta|q) d_q x$ is the orthogonality measure for little q -Jacobi polynomials. It turns out that in order to have formulas and identities similar to those of the usual little q -Jacobi polynomials one needs to keep one of the parameters α_i or β_i fixed and change the other parameters for the r measures. This gives two kinds of multiple little q -Jacobi polynomials. Note that these multiple little q -Jacobi polynomials should not be confused with multivariable little q -Jacobi polynomials, introduced by Stokman

[13]. In [12] the multiple little q -Jacobi polynomials of the first kind are used to prove some irrationality results for $\zeta_q(1)$ and $\zeta_q(2)$.

2.1 Multiple little q -Jacobi polynomials of the first kind

Multiple little q -Jacobi polynomials of the first kind $p_{\vec{n}}(x; \vec{\alpha}, \beta|q)$ are monic polynomials of degree $|\vec{n}|$ satisfying the orthogonality relations

$$\int_0^1 p_{\vec{n}}(x; \vec{\alpha}, \beta|q) x^k w(x; \alpha_j, \beta|q) d_q x = 0, \quad k = 0, 1, \dots, n_j - 1, \quad j = 1, 2, \dots, r, \quad (2.1)$$

where $\alpha_1, \dots, \alpha_r, \beta > -1$. Observe that all the measures are orthogonality measures for little q -Jacobi polynomials with the same parameter β but with different parameters α_j . All the multi-indices will be normal when we impose the condition that $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$, because then all the measures are absolutely continuous with respect to $w(x; 0, \beta|q) d_q x$ and the system of functions

$$x^{\alpha_1}, x^{\alpha_1+1}, \dots, x^{\alpha_1+n_1-1}, x^{\alpha_2}, x^{\alpha_2+1}, \dots, x^{\alpha_2+n_2-1}, \dots, x^{\alpha_r}, x^{\alpha_r+1}, \dots, x^{\alpha_r+n_r-1}$$

is a Chebyshev system on $(0, 1)$, so that the measures (μ_1, \dots, μ_r) form a so-called AT-system, which implies that all the multi-indices are normal [11, Theorem 4.3].

There are r *raising operations* for these multiple orthogonal polynomials.

Theorem 2.1 *Suppose that $\alpha_1, \dots, \alpha_r, \beta > 0$, with $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$, and put $p = 1/q$, then*

$$\begin{aligned} \mathcal{D}_p [w(x; \alpha_j, \beta|q) p_{\vec{n}}(x; \vec{\alpha}, \beta|q)] \\ = \frac{q^{\alpha_j+\beta+|\vec{n}|} - 1}{(1-q)q^{\alpha_j+|\vec{n}|-1}} w(x; \alpha_j - 1, \beta - 1|q) p_{\vec{n}+\vec{e}_j}(x; \vec{\alpha} - \vec{e}_j, \beta - 1|q), \end{aligned} \quad (2.2)$$

for $1 \leq j \leq r$, where $\vec{e}_1 = (1, 0, 0, \dots, 0), \dots, \vec{e}_r = (0, \dots, 0, 0, 1)$ are the standard unit vectors.

Observe that these operations raise one of the indices in the multi-index and lower the parameter β and one of the components of $\vec{\alpha}$.

Proof: First observe that

$$\begin{aligned} \mathcal{D}_p [w(x; \alpha_j, \beta|q) p_{\vec{n}}(x; \vec{\alpha}, \beta|q)] \\ = w(x; \alpha_j - 1, \beta - 1|q) \frac{(1 - q^\beta x) p_{\vec{n}}(x; \vec{\alpha}, \beta|q) - p^{\alpha_j} (1 - x) p_{\vec{n}}(px; \vec{\alpha}, \beta|q)}{1 - p}, \end{aligned}$$

so that

$$\mathcal{D}_p [w(x; \alpha_j, \beta|q) p_{\vec{n}}(x; \vec{\alpha}, \beta|q)] = -\frac{1 - q^{\alpha_j+\beta+|\vec{n}|}}{(1-q)q^{\alpha_j+|\vec{n}|-1}} w(x; \alpha_j - 1, \beta - 1|q) Q_{|\vec{n}|+1}(x), \quad (2.3)$$

where $Q_{|\vec{n}|+1}$ is a monic polynomial of degree $|\vec{n}| + 1$. We will show that this monic polynomial $Q_{|\vec{n}|+1}$ satisfies the multiple orthogonality conditions (2.1) of $p_{\vec{n}+\vec{e}_j}(x; \vec{\alpha} - \vec{e}_j, \beta - 1|q)$ and hence, since all $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$, the unicity of the multiple orthogonal polynomials implies that $Q_{|\vec{n}|+1}(x) = p_{\vec{n}+\vec{e}_j}(x; \vec{\alpha} - \vec{e}_j, \beta - 1|q)$. Integration by parts for the q -integral is given by the rule

$$\int_0^1 f(x) \mathcal{D}_p g(x) = -q \int_0^1 g(x) \mathcal{D}_q f(x), \quad \text{if } g(p) = 0. \quad (2.4)$$

If we apply this, then

$$\begin{aligned} \frac{1 - q^{\alpha_j + \beta + |\vec{n}|}}{(1 - q)q^{\alpha_j + |\vec{n}| - 1}} \int_0^1 x^k w(x; \alpha_j - 1, \beta - 1|q) Q_{|\vec{n}|+1}(x) d_q x \\ = -q \int_0^1 w(x; \alpha_j, \beta|q) p_{\vec{n}}(x; \vec{\alpha}, \beta|q) \mathcal{D}_q x^k d_q x, \end{aligned}$$

and since

$$\mathcal{D}_q x^k = \begin{cases} \frac{1 - q^k}{1 - q} x^{k-1} & \text{if } k \geq 1, \\ 0 & \text{if } k = 0, \end{cases}$$

we find that

$$\int_0^1 x^k w(x; \alpha_j - 1, \beta - 1|q) Q_{|\vec{n}|+1}(x) d_q x = 0, \quad k = 0, 1, \dots, n_j.$$

For the other components α_i ($i \neq j$) of $\vec{\alpha}$ we have

$$\begin{aligned} \frac{1 - q^{\alpha_j + \beta + |\vec{n}|}}{(1 - q)q^{\alpha_j + |\vec{n}| - 1}} \int_0^1 x^k w(x; \alpha_i, \beta - 1|q) Q_{|\vec{n}|+1}(x) d_q x \\ = \frac{1 - q^{\alpha_j + \beta + |\vec{n}|}}{(1 - q)q^{\alpha_j + |\vec{n}| - 1}} \int_0^1 x^{k + \alpha_i - \alpha_j + 1} w(x; \alpha_j - 1, \beta - 1|q) Q_{|\vec{n}|+1}(x) d_q x \\ = -q \int_0^1 w(x; \alpha_j, \beta|q) p_{\vec{n}}(x; \vec{\alpha}, \beta|q) \mathcal{D}_q x^{k + \alpha_i - \alpha_j + 1} d_q x, \end{aligned}$$

and since $\alpha_i - \alpha_j \notin \mathbb{Z}$ we have

$$\mathcal{D}_q x^{k + \alpha_i - \alpha_j + 1} = \frac{1 - q^{k + \alpha_i - \alpha_j}}{1 - q} x^{k + \alpha_i - \alpha_j},$$

hence

$$\int_0^1 x^k w(x; \alpha_i, \beta - 1|q) Q_{|\vec{n}|+1}(x) d_q x = 0, \quad k = 0, 1, \dots, n_i - 1.$$

Hence all the orthogonality conditions for $p_{\vec{n}+\vec{e}_j}(x; \vec{\alpha} - \vec{e}_j, \beta - 1|q)$ are indeed satisfied. \square

As a consequence we find a *Rodrigues formula*:

Theorem 2.2 *The multiple little q -Jacobi polynomials of the first kind are given by*

$$p_{\vec{n}}(x; \vec{\alpha}, \beta|q) = C(\vec{n}, \vec{\alpha}, \beta) \frac{(q^{\beta+1}x; q)_{\infty}}{(qx; q)_{\infty}} \prod_{j=1}^r \left(x^{-\alpha_j} \mathcal{D}_p^{n_j} x^{\alpha_j+n_j} \right) \frac{(qx; q)_{\infty}}{(q^{\beta+|\vec{n}|+1}x; q)_{\infty}}, \quad (2.5)$$

where the product of the difference operators can be taken in any order and

$$C(\vec{n}, \vec{\alpha}, \beta) = (-1)^{|\vec{n}|} \frac{(1-q)^{|\vec{n}|} q^{\sum_{j=1}^r (\alpha_j-1)n_j + \sum_{1 \leq j \leq k \leq r} n_j n_k}}{\prod_{j=1}^r (q^{\alpha_j+\beta+|\vec{n}|+1}; q)_{n_j}}.$$

Proof: If we apply the raising operator for α_j recursively n_j times, then

$$\begin{aligned} \mathcal{D}_p^{n_j} w(x; \alpha_j, \beta|q) p_{\vec{m}}(x; \vec{\alpha}, \beta|q) &= (-1)^{n_j} \frac{(q^{\alpha_j+\beta+|\vec{m}|-n_j+1}; q)_{n_j}}{(1-q)^{n_j} q^{(\alpha_j+|\vec{m}|-1)n_j}} \\ &\quad \times w(x; \alpha_j - n_j, \beta - n_j|q) p_{\vec{m}+n_j \vec{e}_j}(x; \vec{\alpha} - n_j \vec{e}_j, \beta - n_j|q). \end{aligned} \quad (2.6)$$

Use this expression with $\vec{m} = \vec{0}$ and $j = 1$, then

$$\begin{aligned} \mathcal{D}_p^{n_1} w(x; \alpha_1, \beta|q) &= (-1)^{n_1} \frac{(q^{\alpha_1+\beta-n_1+1}; q)_{n_1}}{(1-q)^{n_1} q^{(\alpha_1-1)n_1}} \\ &\quad \times w(x; \alpha_1 - n_1, \beta - n_1|q) p_{n_1 \vec{e}_1}(x; \vec{\alpha} - n_1 \vec{e}_1, \beta - n_1|q). \end{aligned}$$

Multiply both sides by $w(x; \alpha_2, \beta - n_1|q)$ and divide by $w(x; \alpha_1 - n_1, \beta - n_1|q)$, then

$$\begin{aligned} x^{n_1+\alpha_2-\alpha_1} \mathcal{D}_p^{n_1} w(x; \alpha_1, \beta|q) &= (-1)^{n_1} \frac{(q^{\alpha_1+\beta-n_1+1}; q)_{n_1}}{(1-q)^{n_1} q^{(\alpha_1-1)n_1}} \\ &\quad \times w(x; \alpha_2, \beta - n_1|q) p_{n_1 \vec{e}_1}(x; \vec{\alpha} - n_1 \vec{e}_1, \beta - n_1|q). \end{aligned}$$

Apply (2.6) with $j = 2$, then

$$\begin{aligned} \mathcal{D}_p^{n_2} x^{n_1+\alpha_2-\alpha_1} \mathcal{D}_p^{n_1} w(x; \alpha_1, \beta|q) &= (-1)^{n_1+n_2} \frac{(q^{\alpha_1+\beta-n_1+1}; q)_{n_1} (q^{\alpha_2+\beta-n_2+1}; q)_{n_2}}{(1-q)^{n_1+n_2} q^{(\alpha_1-1)n_1 + (\alpha_2-1+n_1)n_2}} \\ &\quad \times w(x; \alpha_2 - n_2, \beta - n_1 - n_2|q) p_{n_1 \vec{e}_1 + n_2 \vec{e}_2}(x; \vec{\alpha} - n_1 \vec{e}_1 - n_2 \vec{e}_2, \beta - n_1 - n_2|q). \end{aligned}$$

Continuing this way we arrive at

$$\begin{aligned} &(\mathcal{D}_p^{n_r} x^{\alpha_r}) \left(x^{n_{r-1}-\alpha_{r-1}} \mathcal{D}_p^{n_{r-1}} x^{\alpha_{r-1}} \right) \cdots \left(x^{n_1-\alpha_1} \mathcal{D}_p^{n_1} \right) w(x; \alpha_1, \beta|q) \\ &= \frac{(-1)^{|\vec{n}|} \prod_{j=1}^r (q^{\alpha_j+\beta-n_j+1}; q)_{n_j}}{(1-q)^{|\vec{n}|} q^{\sum_{j=1}^r (\alpha_j-1)n_j + \sum_{1 \leq j < k \leq r} n_j n_k}} w(x; \alpha_r - n_r, \beta - |\vec{n}||q) p_{\vec{n}}(x; \vec{\alpha} - \vec{n}, \beta - |\vec{n}||q). \end{aligned}$$

Now replace each α_j by $\alpha_j + n_j$ and β by $\beta + |\vec{n}|$, then the required expression follows. The order in which we took the raising operators is irrelevant. \square

We can obtain an explicit expression of the multiple little q -Jacobi polynomials of the first kind using this Rodrigues formula. Indeed, if we use the q -binomial theorem, then

$$\frac{(qx; q)_\infty}{(q^{\beta+|\vec{n}|+1}x; q)_\infty} = \sum_{k=0}^{\infty} \frac{(q^{-\beta-|\vec{n}|}; q)_k}{(q; q)_k} q^{(\beta+|\vec{n}|+1)k} x^k.$$

Use this in (2.5), together with

$$x^{-\alpha} \mathcal{D}_p^n x^{\alpha+n+k} = \frac{(q^{\alpha+1}; q)_n}{(1-q)^n} \frac{(q^{\alpha+n+1}; q)_k}{(q^{\alpha+1}; q)_k} q^{-n(k+\alpha)-n(n-1)/2} x^k,$$

then this gives

$$p_{\vec{n}}(x; \vec{\alpha}, \beta | q) = C(\vec{n}, \vec{\alpha}, \beta) \frac{\prod_{j=1}^r (q^{\alpha_j+1}; q)_{n_j}}{(1-q)^{|\vec{n}|}} q^{-\sum_{j=1}^r \alpha_j n_j - \sum_{j=1}^r \binom{n_j}{2}} \frac{(q^{\beta+1}x; q)_\infty}{(qx; q)_\infty} {}_{r+1}\phi_r \left(\begin{matrix} q^{-\beta-|\vec{n}|}, q^{\alpha_1+n_1+1}, \dots, q^{\alpha_r+n_r+1} \\ q^{\alpha_1+1}, \dots, q^{\alpha_r+1} \end{matrix} \middle| q; q^{\beta+1}x \right). \quad (2.7)$$

This explicit expression uses a non-terminating basic hypergeometric series, except when β is an integer. Another representation, using only finite sums, can be obtained by using the Rodrigues formula (1.8) r times. For $r = 2$ this gives

Theorem 2.3 *The multiple little q -Jacobi polynomials of the first kind (for $r = 2$) are given by*

$$p_{n,m}(x; (\alpha_1, \alpha_2), \beta | q) = \frac{q^{nm+m^2+n^2+\alpha_1 n+\alpha_2 m} (q^{-\alpha_1-n}; q)_n (q^{-\alpha_2-m}; q)_m}{(q^{\alpha_1+\beta+n+m+1}; q)_n (q^{\alpha_2+\beta+n+m+1}; q)_m} \times \sum_{\ell=0}^n \sum_{k=0}^m \frac{(q^{-n}; q)_\ell (q^{-m}; q)_k (q^{\alpha_2+\beta+m+n+1}; q)_k (q^{\alpha_1+\beta+n+1}; q)_{k+\ell} (q^{\alpha_1+n+1}; q)_k}{(q^{\alpha_2+1}; q)_k (q^{\alpha_1+1}; q)_{k+\ell} (q^{\alpha_1+\beta+n+1}; q)_k} \times \frac{q^{k+\ell} x^{k+\ell}}{q^{kn} (q; q)_k (q; q)_\ell}. \quad (2.8)$$

Proof: For $r = 2$ the Rodrigues formula (2.5) is

$$p_{n,m}(x; (\alpha_1, \alpha_2), \beta | q) = \frac{(-1)^{n+m} (1-q)^{n+m} q^{\alpha_1 n+\alpha_2 m-n-m+nm+n^2+m^2}}{(q^{\alpha_1+\beta+n+m+1}; q)_n (q^{\alpha_2+\beta+n+m+1}; q)_m} \times \frac{(q^{\beta+1}x; q)_\infty}{(qx; q)_\infty} x^{-\alpha_1} \mathcal{D}_p^n x^{\alpha_1+n-\alpha_2} \mathcal{D}_p^m x^{\alpha_2+m} \frac{(qx; q)_\infty}{(q^{\beta+n+m+1}x; q)_\infty}.$$

Observe that by the Rodrigues formula (1.8) for the little q -Jacobi polynomials

$$\begin{aligned} \mathcal{D}_p^m x^{\alpha_2+m} \frac{(qx; q)_\infty}{(q^{\beta+n+m+1}x; q)_\infty} \\ = \frac{(-1)^m (q^{\alpha_2+\beta+n+m+1}; q)_m}{(1-q)^m q^{\alpha_2 m+m^2-m}} \frac{(qx; q)_\infty}{(q^{\beta+n+1}x; q)_\infty} x^{\alpha_2} p_m(x; \alpha_2, \beta+n | q), \end{aligned}$$

and hence

$$p_{n,m}(x; (\alpha_1, \alpha_2), \beta|q) = \frac{(-1)^n(1-q)^n q^{\alpha_1 n - n + nm + n^2}}{(q^{\alpha_1 + \beta + n + m + 1}; q)_n} \\ \times \frac{(q^{\beta+1}x; q)_\infty}{(qx; q)_\infty} x^{-\alpha_1} \mathcal{D}_p^n x^{\alpha_1 + n} \frac{(qx; q)_\infty}{(q^{\beta+n+1}x; q)_\infty} p_m(x; \alpha_2, \beta + n|q).$$

Now use the explicit expression (1.9) to find

$$p_{n,m}(x; (\alpha_1, \alpha_2), \beta|q) = \frac{(-1)^n(1-q)^n q^{\alpha_1 n + \alpha_2 m - n + nm + n^2 + m^2} (q^{-m-\alpha_2}; q)_m}{(q^{\alpha_1 + \beta + n + m + 1}; q)_n (q^{\alpha_2 + \beta + n + m + 1}; q)_m} \\ \times \frac{(q^{\beta+1}x; q)_\infty}{(qx; q)_\infty} x^{-\alpha_1} \sum_{k=0}^m \frac{(q^{-m}; q)_k (q^{\alpha_2 + \beta + n + m + 1}; q)_k q^k}{(q^{\alpha_2+1}; q)_k (q; q)_k} \mathcal{D}_p^n x^{\alpha_1 + n + k} \frac{(qx; q)_\infty}{(q^{\beta+n+1}x; q)_\infty}.$$

In this expression we recognize

$$\mathcal{D}_p^n x^{\alpha_1 + n + k} \frac{(qx; q)_\infty}{(q^{\beta+n+1}x; q)_\infty} \\ = \frac{(-1)^n (q^{\alpha_1 + \beta + k + n + 1}; q)_n}{(1-q)^n q^{\alpha_1 n + kn + n^2 - n}} x^{\alpha_1 + k} \frac{(qx; q)_\infty}{(q^{\beta+1}x; q)_\infty} p_n(x; \alpha_1 + k, \beta|q),$$

hence

$$p_{n,m}(x; (\alpha_1, \alpha_2), \beta|q) = \frac{q^{\alpha_2 m + nm + m^2} (q^{-m-\alpha_2}; q)_m}{(q^{\alpha_1 + \beta + n + m + 1}; q)_n (q^{\alpha_2 + \beta + n + m + 1}; q)_m} \\ \times \sum_{k=0}^m \frac{(q^{-m}; q)_k (q^{\alpha_2 + \beta + n + m + 1}; q)_k (q^{\alpha_1 + \beta + k + n + 1}; q)_n q^k}{(q^{\alpha_2+1}; q)_k (q; q)_k q^{kn}} x^k p_n(x; \alpha_1 + k, \beta|q).$$

If we use the explicit expression (1.9) for the little q -Jacobi polynomials once more, then after some simplifications we finally arrive at (2.8). \square

2.2 Multiple little q -Jacobi polynomials of the second kind

Multiple little q -Jacobi polynomials of the second kind $p_{\vec{n}}(x; \alpha, \vec{\beta}|q)$ are monic polynomials of degree $|\vec{n}|$ satisfying the orthogonality relations

$$\int_0^1 p_{\vec{n}}(x; \alpha, \vec{\beta}|q) x^k w(x; \alpha, \beta_j|q) d_q x = 0, \quad k = 0, 1, \dots, n_j - 1, \quad j = 1, 2, \dots, r, \quad (2.9)$$

where $\alpha, \beta_1, \dots, \beta_r > -1$. Observe that all the measures are orthogonality measures for little q -Jacobi polynomials with the same parameter α but with different parameters β_j . All the multi-indices will be normal when we impose the condition that $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$, because then all the measures are absolutely continuous with respect to

$(qx; q)_\infty w(x; \alpha, 0|q) d_q x$ and the system of functions

$$\begin{aligned} & \frac{1}{(q^{\beta_1+1}x; q)_\infty}, \frac{x}{(q^{\beta_1+1}x; q)_\infty}, \dots, \frac{x^{n_1-1}}{(q^{\beta_1+1}x; q)_\infty}, \frac{1}{(q^{\beta_2+1}x; q)_\infty}, \frac{x}{(q^{\beta_2+1}x; q)_\infty}, \\ & \dots, \frac{x^{n_2-1}}{(q^{\beta_2+1}x; q)_\infty}, \dots, \frac{1}{(q^{\beta_r+1}x; q)_\infty}, \frac{x}{(q^{\beta_r+1}x; q)_\infty}, \dots, \frac{x^{n_r-1}}{(q^{\beta_r+1}x; q)_\infty} \end{aligned}$$

is a Chebyshev system¹ on $[0, 1]$, so that the vector of measures (μ_1, \dots, μ_r) forms an AT-system, which implies that all the multi-indices are normal [11, Theorem 4.3].

Again there are r raising operations

Theorem 2.4 Suppose that $\alpha, \beta_1, \dots, \beta_r > 0$, with $\beta_i - \beta_j \notin \mathbb{Z}$ when $i \neq j$, and put $p = 1/q$, then

$$\begin{aligned} \mathcal{D}_p \left[w(x; \alpha, \beta_j|q) p_{\vec{n}}(x; \alpha, \vec{\beta}|q) \right] \\ = \frac{q^{\alpha+\beta_j+|\vec{n}|} - 1}{(1-q)q^{\alpha+|\vec{n}|-1}} w(x; \alpha-1, \beta_j-1|q) p_{\vec{n}+\vec{e}_j}(x; \alpha-1, \vec{\beta}-\vec{e}_j|q), \end{aligned} \quad (2.10)$$

for $1 \leq j \leq r$, where $\vec{e}_1 = (1, 0, 0, \dots, 0), \dots, \vec{e}_r = (0, \dots, 0, 0, 1)$ are the standard unit vectors.

Observe that these operations raise one of the indices in the multi-index and lower the parameter α and one of the components of $\vec{\beta}$.

Proof: Again we see that

$$\mathcal{D}_p \left[w(x; \alpha, \beta_j|q) p_{\vec{n}}(x; \alpha, \vec{\beta}|q) \right] = \frac{q^{\alpha+\beta_j+|\vec{n}|} - 1}{(1-q)q^{\alpha+|\vec{n}|-1}} w(x; \alpha-1, \beta_j-1|q) Q_{|\vec{n}|+1}(x), \quad (2.11)$$

where $Q_{|\vec{n}|+1}$ is a monic polynomial of degree $|\vec{n}| + 1$. We will show that this monic polynomial $Q_{|\vec{n}|+1}$ satisfies the multiple orthogonality conditions (2.9) of $p_{\vec{n}+\vec{e}_j}(x; \alpha-1, \vec{\beta}-\vec{e}_j|q)$ and hence, since all $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$, the unicity of the multiple orthogonal polynomials implies that $Q_{|\vec{n}|+1}(x) = p_{\vec{n}+\vec{e}_j}(x; \alpha-1, \vec{\beta}-\vec{e}_j|q)$. Integration by parts gives

$$\begin{aligned} & \frac{1 - q^{\alpha+\beta_j+|\vec{n}|}}{(1-q)q^{\alpha+|\vec{n}|-1}} \int_0^1 x^k w(x; \alpha-1, \beta_j-1|q) Q_{|\vec{n}|+1}(x) d_q x \\ & = -q \int_0^1 w(x; \alpha, \beta_j|q) p_{\vec{n}}(x; \alpha, \vec{\beta}|q) \mathcal{D}_q x^k d_q x, \end{aligned}$$

so that

$$\int_0^1 x^k w(x; \alpha-1, \beta_j-1|q) Q_{|\vec{n}|+1}(x) d_q x = 0, \quad k = 0, 1, \dots, n_j.$$

¹ The fact that this system is a Chebyshev system is not obvious but is left as an advanced problem for the reader.

For the other components β_i ($i \neq j$) of $\vec{\beta}$ we have

$$\begin{aligned} & \frac{1 - q^{\alpha + \beta_j + |\vec{n}|}}{(1 - q)q^{\alpha + |\vec{n}| - 1}} \int_0^1 x^k w(x; \alpha - 1, \beta_i | q) Q_{|\vec{n}|+1}(x) d_q x \\ &= \frac{1 - q^{\alpha + \beta_j + |\vec{n}|}}{(1 - q)q^{\alpha + |\vec{n}| - 1}} \int_0^1 x^k \frac{(q^{\beta_j} x; q)_\infty}{(q^{\beta_i+1} x; q)_\infty} w(x; \alpha - 1, \beta_j - 1 | q) Q_{|\vec{n}|+1}(x) d_q x \\ &= -q \int_0^1 w(x; \alpha, \beta_j | q) p_{\vec{n}}(x; \alpha, \vec{\beta} | q) \mathcal{D}_q \left(x^k \frac{(q^{\beta_j} x; q)_\infty}{(q^{\beta_i+1} x; q)_\infty} \right) d_q x, \end{aligned}$$

and since $\beta_i - \beta_j \notin \mathbb{Z}$ we have

$$\mathcal{D}_q \left(x^k \frac{(q^{\beta_j} x; q)_\infty}{(q^{\beta_i+1} x; q)_\infty} \right) = x^{k-1} \frac{(q^{\beta_j+1} x; q)_\infty}{(q^{\beta_i+1} x; q)_\infty} a_k(x),$$

where each a_k is a polynomial of degree exactly 1 and $a_0(0) = 0$. Therefore

$$\int_0^1 x^k w(x; \alpha - 1, \beta_i | q) Q_{|\vec{n}|+1}(x) d_q x = 0, \quad k = 0, 1, \dots, n_i - 1.$$

Hence all the orthogonality conditions for $p_{\vec{n} + \vec{e}_j}(x; \alpha - 1, \vec{\beta} - \vec{e}_j | q)$ are indeed satisfied. \square

As a consequence we again find a *Rodrigues formula*:

Theorem 2.5 *The multiple little q -Jacobi polynomials of the second kind are given by*

$$p_{\vec{n}}(x; \alpha, \vec{\beta} | q) = \frac{C(\vec{n}, \alpha, \vec{\beta})}{(qx; q)_\infty x^\alpha} \prod_{j=1}^r \left((q^{\beta_j+1} x; q)_\infty \mathcal{D}_p^{n_j} \frac{1}{(q^{\beta_j+n_j+1} x; q)_\infty} \right) (qx; q)_\infty x^{\alpha + |\vec{n}|}, \quad (2.12)$$

where the product of the difference operators can be taken in any order and

$$C(\vec{n}, \alpha, \vec{\beta}) = (-1)^{|\vec{n}|} \frac{(1 - q)^{|\vec{n}|} q^{(\alpha + |\vec{n}| - 1)|\vec{n}|}}{\prod_{j=1}^r (q^{\alpha + \beta_j + |\vec{n}| + 1}; q)_{n_j}}.$$

Proof: The proof can be given in a similar way as in the case of little q -Jacobi polynomials of the first kind by repeated application of the raising operators. Alternatively one can use induction on r . For $r = 1$ the Rodrigues formula is the same as (1.8). Suppose that the Rodrigues formula (2.12) holds for $r - 1$. Observe that the multiple orthogonal polynomials with multi-index (n_1, \dots, n_{r-1}) for $r - 1$ measures $(\mu_1, \dots, \mu_{r-1})$ coincide with the multiple orthogonal polynomials with multi-index $(n_1, n_2, \dots, n_{r-1}, 0)$ for r measures (μ_1, \dots, μ_r) for any measure μ_r . Use the Rodrigues formula for $r - 1$ for the polynomial $p_{\vec{n} - n_r \vec{e}_r}(x; \alpha + n_r, \vec{\beta} + n_r \vec{e}_r | q)$ to find

$$\begin{aligned} & w(x; \alpha + n_r, \beta_r + n_r | q) p_{\vec{n} - n_r \vec{e}_r}(x; \alpha + n_r, \vec{\beta} + n_r \vec{e}_r | q) = C(\vec{n} - n_r \vec{e}_r, \alpha + n_r, \vec{\beta}) \\ & \times \frac{1}{(q^{\beta_r + n_r + 1} x; q)_\infty} \prod_{j=1}^{r-1} \left((q^{\beta_j+1} x; q)_\infty \mathcal{D}_p^{n_j} \frac{1}{(q^{\beta_j+n_j+1} x; q)_\infty} \right) (qx; q)_\infty x^{\alpha + |\vec{n}|}. \end{aligned}$$

Now apply the raising operation (2.10) for β_r to this expression n_r times to find the required expression. \square

In a similar way as for the first kind multiple little q -Jacobi polynomials we can find an explicit formula with finite sums using the Rodrigues formula for little q -Jacobi polynomials r times. For $r = 2$ this gives the following:

Theorem 2.6 *The multiple little q -Jacobi polynomials of the second kind (for $r = 2$) are explicitly given by*

$$p_{n,m}(x; \alpha, (\beta_1, \beta_2)|q) = \frac{q^{\alpha(n+m)+n^2+m^2+nm}(q^{-m-\alpha}; q)_m (q^{-n-\alpha}; q)_n (q^{\alpha+1}; q)_{m+n}}{(q^{\alpha+\beta_1+n+m+1}; q)_n (q^{\alpha+\beta_2+n+m+1}; q)_m (q^{\alpha+1}; q)_n (q^{\alpha+1}; q)_m} \\ \times \sum_{\ell=0}^n \sum_{k=0}^m \frac{(q^{-n}; q)_\ell (q^{-m}; q)_k (q^{\alpha+\beta_2+n+m+1}; q)_k (q^{\alpha+\beta_1+n+1}; q)_{k+\ell}}{(q^{\alpha+1}; q)_{k+\ell} (q^{\alpha+\beta_1+n+1}; q)_k} \\ \times \frac{q^{k+\ell} x^{k+\ell}}{q^{nk} (q; q)_k (q; q)_\ell}. \quad (2.13)$$

Proof: The Rodrigues formula (2.12) for $r = 2$ becomes

$$p_{n,m}(x; \alpha, (\beta_1, \beta_2)|q) = \frac{(-1)^{n+m} (1-q)^{n+m} q^{(\alpha+n+m-1)(n+m)}}{(q^{\alpha+\beta_1+n+m+1}; q)_n (q^{\alpha+\beta_2+n+m+1}; q)_m} \\ \times x^{-\alpha} \frac{(q^{\beta_1+1}x; q)_\infty}{(qx; q)_\infty} \mathcal{D}_p^n \frac{(q^{\beta_2+1}x; q)_\infty}{(q^{\beta_1+n+1}x; q)_\infty} \mathcal{D}_p^m \frac{(qx; q)_\infty}{(q^{\beta_2+m+1}x; q)_\infty} x^{\alpha+n+m}.$$

The Rodrigues formula (1.8) for little q -Jacobi polynomials gives

$$\mathcal{D}_p^m \frac{(qx; q)_\infty}{(q^{\beta_2+m+1}x; q)_\infty} x^{\alpha+n+m} \\ = \frac{(-1)^m (q^{\alpha+\beta_2+n+m+1}; q)_m}{(1-q)^m q^{\alpha m+m^2-m+n m}} x^{\alpha+n} \frac{(qx; q)_\infty}{(q^{\beta_2+1}x; q)_\infty} p_m(x; \alpha+n, \beta_2|q),$$

hence

$$p_{n,m}(x; \alpha, (\beta_1, \beta_2)|q) = \frac{(-1)^n (1-q)^n q^{\alpha n+n^2+nm-n}}{(q^{\alpha+\beta_1+n+m+1}; q)_n} \\ \times x^{-\alpha} \frac{(q^{\beta_1+1}x; q)_\infty}{(qx; q)_\infty} \mathcal{D}_p^n x^{\alpha+n} \frac{(qx; q)_\infty}{(q^{\beta_1+n+1}x; q)_\infty} p_m(x; \alpha+n, \beta_2|q).$$

Now use the explicit expression (1.9) for the little q -Jacobi polynomials to find

$$p_{n,m}(x; \alpha, (\beta_1, \beta_2)|q) = \frac{(-1)^n (1-q)^n q^{\alpha(n+m)+n^2+m^2+2nm-n} (q^{-m-n-\alpha}; q)_m}{(q^{\alpha+\beta_1+n+m+1}; q)_n (q^{\alpha+\beta_2+n+m+1}; q)_m} \\ \times \frac{(q^{\beta_1+1}x; q)_\infty}{x^\alpha (qx; q)_\infty} \sum_{k=0}^m \frac{(q^{-m}; q)_k (q^{\alpha+\beta_2+n+m+1}; q)_k q^k}{(q^{\alpha+n+1}; q)_k (q; q)_k} \mathcal{D}_p^n x^{\alpha+n+k} \frac{(qx; q)_\infty}{(q^{\beta_1+n+1}x; q)_\infty}.$$

Again we recognize a little q -Jacobi polynomial

$$\begin{aligned} \mathcal{D}_p^n x^{\alpha+n+k} \frac{(qx; q)_\infty}{(q^{\beta_1+n+1}x; q)_\infty} \\ = \frac{(-1)^n (q^{\alpha+\beta_1+k+n+1}; q)_n}{(1-q)^n q^{\alpha n + kn + n^2 - n}} x^{\alpha+k} \frac{(qx; q)_\infty}{(q^{\beta_1+1}x; q)_\infty} p_n(x; \alpha + k, \beta_1 | q), \end{aligned}$$

and if we use the explicit expression (1.9) for this little q -Jacobi polynomial, then we find (2.13) after some simplifications. \square

3 Zeros

The zeros of the multiple little q -Jacobi polynomials (first and second kind) are all real, simple and in the interval $(0, 1)$. This is a consequence of the fact that μ_1, \dots, μ_r form an AT-system [11, first Corollary on p. 141]. For the usual orthogonal polynomials with positive orthogonality measure μ we know that an interval $[c, d]$ for which the orthogonality measure has no mass, i.e., $\mu([c, d]) = 0$, can have at most one zero of each orthogonal polynomial p_n . In particular this means that each orthogonal polynomial p_n on the exponential lattice $\{q^k, k = 0, 1, 2, \dots\}$ can have at most one zero between two points q^{k+1} and q^k of the lattice. A similar result holds for multiple orthogonal polynomials if we impose some conditions on the measures μ_i .

Theorem 3.1 *Suppose μ_1, \dots, μ_r are positive measures on $[a, b]$ with infinitely many points in their support, which form an AT system, i.e., μ_k is absolutely continuous with respect to μ_1 for $2 \leq k \leq r$ with*

$$\frac{d\mu_k(x)}{d\mu_1(x)} = w_k(x),$$

and

$$1, x, \dots, x^{n_1-1}, w_2(x), xw_2(x), \dots, x^{n_2-1}w_2(x), \dots, w_r(x), xw_r(x), \dots, x^{n_r-1}w_r(x)$$

are a Chebyshev system on $[a, b]$ for every multi-index \vec{n} . If $[c, d]$ is an interval such that $\mu_1([c, d]) = 0$, then each multiple orthogonal polynomial $p_{\vec{n}}$ has at most one zero in $[c, d]$.

Proof: Suppose that $p_{\vec{n}}$ is a multiple orthogonal polynomial with two zeros x_1 and x_2 in $[c, d]$. We can then write it as $p_{\vec{n}}(x) = (x-x_1)(x-x_2)q_{|\vec{n}|-2}(x)$, where $q_{|\vec{n}|-2}$ is a polynomial of degree $|\vec{n}|-2$. Consider a function $A(x) = \sum_{j=1}^r A_j(x)w_j(x)$, where $w_1 = 1$ and each A_j is a polynomial of degree $m_j - 1 \leq n_j - 1$, with $|\vec{m}| = |\vec{n}| - 1$. Since we are dealing with a Chebyshev system, there is a unique function A satisfying the interpolation conditions

$$A(y) = \begin{cases} 0 & \text{if } y \text{ is a zero of } q_{|\vec{n}|-2}, \\ 1 & \text{if } y = x_1. \end{cases}$$

Furthermore A has $|\vec{n}| - 2$ zeros in $[a, b]$ and these are the only sign changes on $[a, b]$. Hence

$$\int_a^b p_{\vec{n}}(x)A(x) d\mu_1(x) = \int_{[a,b] \setminus [c,d]} (x - x_1)(x - x_2)q_{|\vec{n}|-2}(x)A(x) d\mu_1(x) \neq 0,$$

since the integrand does not change sign on $[a, b] \setminus [c, d]$. On the other hand

$$\int_a^b p_{\vec{n}}(x)A(x) d\mu_1(x) = \sum_{j=1}^r \int_a^b p_{\vec{n}}(x)A_j(x) d\mu_j(x) = 0,$$

since every term in the sum vanishes because of the orthogonality conditions. This contradiction implies that $p_{\vec{n}}$ can't have two zeros in $[c, d]$. \square

In particular this theorem tells us that the zeros of the multiple little q -Jacobi polynomials are always separated by the points q^k and that between two points q^{k+1} and q^k there can be at most one zero of a multiple little q -Jacobi polynomials. Note that the points q^k have one accumulation point at 0, hence as a consequence the zeros of the multiple little q -Jacobi polynomials (first and second kind) accumulate at the origin.

4 Asymptotic behavior

The asymptotic behavior of little q -Jacobi polynomials was given by Ismail and Wilson [7] and an asymptotic expansion was given by Ismail [6]. In this section we give the asymptotic behavior of the multiple little q -Jacobi polynomials which extends the result of Ismail and Wilson.

Theorem 4.1 *For the multiple little q -Jacobi polynomials of the first kind we have*

$$\lim_{n,m \rightarrow \infty} x^{n+m} p_{n,m}(1/x; (\alpha_1, \alpha_2), \beta|q) = (x; q)_{\infty}. \quad (4.1)$$

The order in which the limits for n and m are taken is irrelevant.

Proof: If we use (2.8) and reverse the order of summation (i.e., change variables $m - k = j$ and $n - \ell = i$), then

$$\begin{aligned} x^{n+m} p_{n,m}(1/x; (\alpha_1, \alpha_2), \beta|q) &= \frac{q^{nm+m^2+n^2+\alpha_1 n+\alpha_2 m}(q^{-\alpha_1-n}; q)_n (q^{-\alpha_2-m}; q)_m}{(q^{\alpha_1+\beta+n+m+1}; q)_n (q^{\alpha_2+\beta+n+m+1}; q)_m} \\ &\times \sum_{i=0}^n \sum_{j=0}^m \frac{(q^{-n}; q)_{n-i} (q^{-m}; q)_{m-j} (q^{\alpha_2+\beta+m+n+1}; q)_{m-j} (q^{\alpha_1+\beta+n+1}; q)_{m+n-i-j} (q^{\alpha_1+n+1}; q)_{m-j}}{(q^{\alpha_2+1}; q)_{m-j} (q^{\alpha_1+1}; q)_{m+n-i-j} (q^{\alpha_1+\beta+n+1}; q)_{m-j}} \\ &\times \frac{q^{m+n-i-j} x^{i+j}}{q^{(m-j)n} (q; q)_{m-j} (q; q)_{n-i}}. \end{aligned}$$

Now observe that

$$\begin{aligned}
(q^{-m}; q)_{m-j} &= (-1)^{m-j} q^{-\frac{m(m+1)}{2} + \frac{j(j+1)}{2}} \frac{(q; q)_m}{(q; q)_j}, \\
(q^{-m-\alpha}; q)_m &= (-1)^m q^{-m(m+1)/2} q^{-m\alpha} (q^{\alpha+1}; q)_m, \\
(q^{c+n}; q)_m &= \frac{(q^c; q)_{n+m}}{(q^c; q)_n},
\end{aligned}$$

therefore we find

$$\begin{aligned}
x^{n+m} p_{n,m}(1/x; (\alpha_1, \alpha_2), \beta | q) &= \frac{(q^{\alpha_2+1}; q)_m (q^{\alpha_1+\beta+1}; q)_{n+m} (q; q)_m (q; q)_n}{(q^{\alpha_1+\beta+1}; q)_{2n+m} (q^{\alpha_2+\beta+1}; q)_{n+2m}} \\
&\times \sum_{i=0}^n \sum_{j=0}^m \frac{(q^{\alpha_2+\beta+1}; q)_{n+2m-j} (q^{\alpha_1+\beta+1}; q)_{2n+m-i-j} (q^{\alpha_1+1}; q)_{n+m-j}}{(q^{\alpha_2+1}; q)_{m-j} (q^{\alpha_1+1}; q)_{m+n-i-j} (q^{\alpha_1+\beta+1}; q)_{m+n-j} (q; q)_{n-i} (q; q)_{m-j}} \\
&\times (-1)^{i+j} q^{\binom{i}{2} + \binom{j}{2}} \frac{q^{nj} x^{i+j}}{(q; q)_i (q; q)_j}.
\end{aligned}$$

If we use Lebesgue's dominated convergence theorem, then we take $n, m \rightarrow \infty$ in each term of the sum. The factor q^{nj} tends to zero whenever $j > 0$, hence the only contributions come from $j = 0$, and we find

$$\lim_{n, m \rightarrow \infty} x^{n+m} p_{n,m}(1/x; (\alpha_1, \alpha_2), \beta | q) = \sum_{i=0}^{\infty} q^{\binom{i}{2}} \frac{(-x)^i}{(q; q)_i}.$$

The right hand side is the q -exponential function

$$E_q(-x) = (x, q)_{\infty},$$

[5, (II.2) in Appendix II], which gives the required result. \square

Theorem 4.2 *For the multiple little q -Jacobi polynomials of the second kind we have*

$$\lim_{n, m \rightarrow \infty} x^{n+m} p_{n,m}(1/x; \alpha, (\beta_1, \beta_2) | q) = (x; q)_{\infty}. \quad (4.2)$$

The order in which the limits for n and m are taken is irrelevant.

Proof: The proof is similar to the case of the first kind multiple little q -Jacobi polynomials, except that now we use the expression (2.13). \square

As a consequence (using Hurwitz' theorem) we see that every zero of $(1/x; q)_{\infty}$, i.e., each number q^k , $k = 0, 1, 2, \dots$, is an accumulation point of zeros of the multiple little q -Jacobi polynomial $p_{n,m}$ of the first and of the second kind.

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